ON MAXIMAL IDEALS IN TWO ALGEBRAS OF OPERATOR VALUED ANALYTIC FUNCTIONS

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ABSTRACT

The maximal ideals in two algebras of operator valued analytic functions in the unit disc are described.

Let N be an n-dimensional Hilbert space, where $\dim N = n < \infty$. We denote by $H^\infty(N\ N)$ the Banach algebra of all norm bounded N-operator valued analytic functions in the open unit disc with the norm defined by

$$\| A \|_\infty = \operatorname*{Sup}_{|z|<1} \| A(z) \|$$

By $A(N, N)$ we denote the subalgebra of all functions having a continuous extension to the closed unit disc. H^∞ and A will denote the corresponding scalar algebras.

It is expected that the maximal ideal structure of these algebras reflects the maximal ideal structure of the corresponding scalar algebras as well as that of $B(N)$ the algebra of all (bounded) linear operators on N, and this is confirmed. Proofs will be given only for the algebra $H^\infty(N, N)$. The corresponding theorems and proofs for $A(N, N)$ differ only slightly by being somewhat simpler as all complex homomorphisms of the scalar algebra rise from evaluation at points of the closed unit disc.

By $\mathscr{M}(H^\infty)$ we denote the maximal ideal space of H^∞. For each element ϕ of $\mathscr{M}(H^\infty)$, ϕ will also denote the corresponding complex homomorphism of H^∞. H^∞ we have a natural involution given by $f \to \tilde{f}$ where $\tilde{f}(z) = \overline{f(\bar{z})}$. This generalizes to $H^\infty(N, N)$ by $\tilde{A}(z) = A(\bar{z})^*$. The involution in H^∞ is reflected by a conjugation in the maximal ideal space. For $\phi \in \mathscr{M}(H^\infty)$ we define $\bar{\phi}$ by $\bar{\phi}(f) = \overline{\phi(\tilde{f})}$. Obviously $\bar{\phi} \in \mathscr{M}(H^\infty)$ and $\bar{\bar{\phi}} = \phi$.

For $\phi \in \mathscr{M}(H^\infty)$, $A \in H^\infty(N, N)$ $\phi(A(z)x, y)$ defines a bounded bilinear form

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in N. Thus there exists a unique linear operator in N, which we denote by $\Phi_\phi(A)$ such that

$$\phi(A(z)x, y) = (\Phi_\phi(A)x, y)$$

THEOREM 1. (i) *Φ_ϕ is a norm decreasing homomorphism of $H^\infty(N\ N)$ onto $B(N)$.*

(ii) $\Phi_\phi(\tilde{A}) = \Phi_{\bar{\phi}}(A)^*$.

PROOF. (i) The linearity of the map Φ_ϕ is obvious. For a constant operator valued function C it is trivial to check that $\Phi_\phi(C) = C$ and similarly for $f \in H^\infty$ and C a constant operator we have $\Phi_\phi(fC) = \phi(f)C$.

Now let $\{e_i \mid i = 1, \cdots, n\}$ be an o.n. basis in N. Let E_{ij} be the linear operator that is defined by

$$E_{ij}e_k = \delta_{jk}e_i, \; k = 1, \cdots, n, \text{ and let } a_{ij} = (Ae_j, e_i).$$

Then clearly $A \in H^\infty(N, N)$ can be written as $\sum_{i,j=1}^n a_{ij}E_{ij}$ and $a_{ij} \in H^\infty$. If $B = \sum_{p,q=1}^n b_{pq}E_{pq}$ then

$$AB = \sum_{i,j=1}^n \sum_{p,q=1}^n a_{ij}b_{pq}E_{ij}E_{pq}$$

$$= \sum_{i,q=1}^n \left\{ \sum_{j=1}^n a_{ij}b_{jq} \right\} E_{iq}$$

as $E_{ij}E_{pq} = \delta_{pj}E_{iq}$

Therefore by linearity we have

$$\Phi_\phi(AB) = \sum_{i,q=1}^n \sum_{j=1}^n \Phi_\phi(a_{ij}b_{jq}E_{iq})$$

$$= \sum_{i,q=1}^n \sum_{j=1}^n \phi(a_{ij}b_{jq})E_{iq} = \sum_{i,q=1}^n \sum_{j=1}^n \phi(a_{ij})\phi(b_{jq})E_{iq}$$

$$= \Phi_\phi(A)\Phi_\phi(B)$$

Here we used the fact that ϕ is a homomorphism of H^∞.

Now we note that

$$\left|(\Phi_\phi(A)x, y)\right| = \left|\phi(Ax, y)\right| \leq \|\phi\| \cdot \operatorname{Sup}_{|z|<1} \left|(A(z)x, y)\right|$$

$$\leqq \operatorname{Sup} \| A(z) \| \cdot \| x \| y \cdot \| \| = \| A \|_\infty \cdot \| x \| \cdot \| y \|.$$

Hence $\| \Phi_\phi(A) \| = \operatorname{Sup}_{\|x\|=\|y\|=1} \left|(\Phi_\phi(A)x, y)\right| \leqq \| A \|_\infty$. It is clear however that $\| \Phi_\phi \| = 1$.

(ii)
$$(\Phi_\phi(\tilde{A})x, y) = \phi(\tilde{A}(z)x, y) = \phi(A(\bar{z})^*x, y)$$
$$= \phi(x, A(\bar{z})y) = \bar{\phi}\overline{(x, A(z)y)} = \bar{\phi}\overline{(A(z)y, x)}$$
$$= (\Phi_{\bar{\phi}}(A)y, x) = (\Phi_{\bar{\phi}}(A)^*x, y).$$

If we consider for points λ in the open unit disc the homomorphism ϕ_λ of evaluation at λ then

$$(\Phi_{\phi_\lambda}(A)x, y) = \phi_\lambda(Ax, y) = (A(\lambda)x, y)$$

as $\Phi_{\phi_\lambda}(A) = A(\lambda)$. Therefore it seems natural to write $A(\phi) = \Phi_\phi(A)$ for every $\phi \in \mathscr{M}(H^\infty)$.

Given an invertible element R of $B(N)$ the map $\Phi_{\phi,R}$ of $H^\infty(N, N)$ onto $B(N)$ is defined by $\Phi_{\phi,R}(A) = R^{-1}\Phi_\phi(A)R$. Obviously $\Phi_{\phi,R}$ is also a homomorphism, though its norm is generally larger than one.

The following is a converse to Theorem 1.

THEOREM 2. *Let Φ be a non-trivial homomorphism of $H^\infty(N, N)$ into $B(N)$ then there exists $\phi \in \mathscr{M}(H^\infty)$ and an invertible element R of $B(N)$ such that $\Phi = \Phi_{\phi,R}$.*

PROOF. $B(N)$ is naturally embedded in $H^\infty(N, N)$ via the constant functions thus Φ restricted to $B(N)$ is a non-trivial homomorphism of $B(N)$ into itself hence it is of the form $\Phi(C) = R^{-1}CR$ for some invertible R in $B(N)$. Correspondingly $f \to fI$ is natural embedding of H^∞ in $H^\infty(N, N)$. Since fI is in the center of $H^\infty(N, N)$ we have for each $A \in H^\infty(N, N)$.

$$\Phi(f)\Phi(A) = \Phi(A)\Phi(f).$$

But as Φ is onto $B(N)$ it follows that $\Phi(f)$ is in the center of $B(N)$, hence of the form $\alpha_f I$. The map $\alpha_f : H^\infty \to C$ is a complex homomorphism of H^∞. Therefore there exists a $\phi \in \mathscr{M}(H^\infty)$ such that $\alpha_f = \phi(f)$. Now as in Theorem 1, $A = \sum_{i,j=1}^n a_{ij}E_{ij}$ implies

$$\Phi(A) = \sum_{i,j=1}^n \Phi(a_{ij})\Phi(E_{ij}) = \sum_{i,j=1}^n \phi(a_{ij})R^{-1}E_{ij}R = R^{-1}\sum_{i,j=1}^n \phi(a_{ij})E_{ij}R$$
$$= R^{-1}\Phi_\phi(A)R = \Phi_{\phi,R}(A).$$

THEOREM 3. *A subset M of $H^\infty(N, N)$ is a maximal left, right or two sided ideal if and only if it is of the form $\{A \mid A(\phi)x = 0\}$, $\{A \mid A(\phi)^*x = o\}$ or*

$\{A \mid A(\phi) = 0\}$ *respectively for some* $\phi \in \mathscr{M}(H^\infty)$ *and a non-zero vector* x *in* N.

PROOF. The maximal left, right and two sided ideals in $B(N)$ are of the form $\{B \mid Bx = 0\}$, $\{B \mid B^*x = 0\}$ and $\{0\}$, therefore their inverse images under the homomorphism $\Phi_{\phi,I}$ are corresponding maximal ideals in $H^\infty(N,N)$.

Conversely let M be a maximal left ideal in $H^\infty(N,N)$ and assume $M \neq \{A \mid A(\phi)x = 0\}$ for all $\phi \in \mathscr{M}(H)^\infty$ and $0 \neq x \in N$. Without loss of generality we may assume $\| x \| = 1$. We will show that $M = H^\infty(N,N)$.

For each pair (ϕ, x), $\phi \in \mathscr{M}(H^\infty)$, $x \in N$ $\| x \| = 1$ there exists an element of M, denoted by $A_{\phi,x}$ such that $A_{\phi,x}(\phi)x \neq 0$. Since $A_{\phi,x}(\psi)t$ is a continuous function of $(\psi, t) \in \mathscr{M}(H^\infty) \times B_N$, where B_N is the unit ball in N, there exists an open neighborhood $U_{\phi,x}$ of (ϕ, x) in the product space $\mathscr{M}(H^\infty) \times B_N$ such that $A_{\phi,x}$ has no zeros in $U_{\phi,x}$. $\{U_{\phi,x} \mid \phi \in \mathscr{M}(H^\infty),\ x \in N \ \| x \| = 1\}$ is an open cover of the compact space $\mathscr{M}(H^\infty) \times B_N$ and thus has a finite subcover $\{U_{\phi_i,x_i} \mid i = 1, \cdots, p\}$ Thus $\sum_{i=1}^p \| A_{\phi_i,x_i}(\psi)x \|$ for all $(\psi, x) \in \mathscr{M}(H^\infty)\, X\, B_N$ By continuity there exist a $\delta > 0$ such that $\sum_{i=1}^p \| A_{\phi_i,x_i}(\psi)x \| \geqq \delta$ and in particular we have $\sum_{i=1}^p \| A_{\phi_i,x_i}(z)x \| \geqq \delta$ for all z in the open unit disc and vectors x of norm one. We invoke now a matrix generalization of the corona theorem [1, Theorem 3.1] which implies the existence of $B_i \in H^\infty(N,N)$ such that

$$\sum_{i=1}^{p} B_i(z) A_{\phi_i,x_i}(z) = I \qquad \text{or } I \in M \qquad \text{and}$$

hence $M = H^\infty(N,N)$. Similarly for right ideals.

For a maximal two sided ideal M, let M_1 be an extension of M to a maximal left ideal which by the foregoing is of the form $M_1 = \{A \mid A(\phi)x = 0\}$. Since M is two sided it follows that $M \subset \{A \mid A(\phi) = 0\}$ and by maximality we must have equality.

REMARK. In proving the corresponding theorem for the algebra $A(N,N)$ we use the following theroem whose proof is exactly analogous to the proof of Theorem 3.1 in [1].

THEOREM 4. a) *Given* $A_i \in A(N,N)$, $i = 1, \cdots, p$ *then a necessary and sufficient condition for the existence of* $B_i \in A(N,N)$ *such that* $\sum_{i=1}^p B_i(z)A_i(z) = I$ *is*

$$\inf \left\{ \sum_{i=1}^{p} \| A_i(z)x \| \;\middle|\; \| x \| = 1, \quad x \in N \right\} > 0$$

for all z *such* $|z| \leqq 1$.

b) *A necessary and sufficient condition for the existence of* $B_i \in A(N,N)$ *such that*

$$\sum_{i=1}^{p} A_i(z) B_i(z) = 1 \quad \textit{is}$$

$$\inf \left\{ \sum_{i=1}^{p} \| A_i(z)^* x \| \quad x \in N, \quad \| x \| = 1 \right\} > 0$$

for all z *such that* $|z| \leqq 1$.

Reference

1. P. A. Fuhrmann, *On the corona theorem and its application to spectral problems in Hilbert space*, Trans. Amer. Math. Soc., **132** (1968), 55–66.

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